# Systems Dynamics In Ramsey Economic Growth Models: The Case for an Ecological Economics Perspective for Economic Modeling

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## 1. Introduction

An expanding body of work in ecological economics shows that the dynamic properties of integrated human-natural systems reflect the complexity of the ecosystem processes upon which they depend (Arrow et. al., 2000; Perrings and Walker, 1997; and Maler, 2000). In this paper, we investigate a complementary question: whether realistic modifications of the economic component of the standard neoclassical growth model can yield similarly diverse outcomes, augmenting the dynamic diversity generated through complex ecological interactions. The conventional economic growth literature (e.g., see Aghion and Howitt, 1998 for a review) and its extensions to natural resources (e.g., see Toman et. al, 1995 for a review) have largely abstracted from the possibility of dynamic complexity.

Our research specifically explores three modifications of the neoclasical economics models. The first study (Krutilla and Reuveny, 2001a) broadens the quality-of-life representation within the conventional economic growth framework to assess whether the broader conception has implications for the system's dynamic behavior. If so, the quality-of-life measure itself introduces another policy-relevant distinction between the terms "growth" and "development" as these have become distinguished in the sustainable development literature (e.g., The Brundtland Commission, 1987; Daly, 1992; World Bank, 1997). This is a modification on the "demand side of the economy" and, as such, can be taken as a representative model of that class of modifications. The second modification is the addition of costly resource extraction in the standard neoclassical growth model (Krutilla and Reuveny, 2001b). Costly resource extraction is a fundamental economic reality in natural resource-dependent economics, and thus is commonly modeled in studies having a sectoral focus on the natural resource industries (e.g., see Clark, 1990; Hartwick and Olewiler, 1986; Munro and Scott, 1985; Clark and Munro, 1982). This is a modification on the supply side of the economy and, as such, can be taken as a representative model of that class of modifications. Finally, the standard Ramsey economic growth model is modified to allow for an endogenous feedback relationship between population growth an income – an empirically realistic specification (Krutilla and Reuveny, 2001c). As a modification of a feedback relationship, this adjustment can be taken as representative model of that class of modifications (others being, for example, the endogenous relationship between pollution emissions and income as suggested in the Kuznets curve literature (e.g., Stern and Commons, 2001).

All of these modifications turn out to inject tremendous dynamic diversity into the system. While the standard economy based on the Ramsey framework exhibits one saddle-point stable stationary state for per capita consumption and capital stock, any of the modifications described above leads to the possibility of multiple steady states in the economy, some of them unstable; multiple balanced growth paths in the presence of technical change; and the possibility of perverse and harmful technology effects.

## 2. The Models

In the standard neoclassical growth model (the "Ramsey Model"), the economy's gross output, Y, is assumed to be produced using two aggregate production inputs, capital, K, and labor, L. The labor force is assumed to be equal to the population and be fully employed. The economy's output is produced using a linearly homogenous production function: Y=F(K,L). F(K,L) is assumed to be twice continuously differentiable, with $F_K > 0$, $F_{KK} < 0$; $F_L > 0$, and $F_{LL} < 0$.

The economy's output is allocated for consumption and capital accumulation, with the equation of motion for capital given by:

$$\frac{dK}{dt} = F(K,L) - \delta K - C. \tag{1}$$

C is aggregate consumption and $\delta$ is the rate of capital depreciation. In the standard Ramsey economic model, the rate of population growth, n, is exogenously given and grows according to the equation of motion: dL/dt=nL.

A representative agent is assumed to have a utility function U(c), where $c \equiv C/L$ is per capita consumption. Utility exhibits the commonly-assumed properties: $U_c>0$, $U_{cc}<0$, $\lim_{c \to 0} (U_c)=\infty$, and $\lim_{c \to \infty} (U_c)=0$. Given a positive discount rate, $\rho$, the representative agent is assumed to solve the following optimal control problem:

$$\begin{aligned}
&\max \int_0^\infty U(c)e^{-\rho t}dt \\
&s.t. \\
&\frac{dK}{dt} = F(K,L) - \delta K - Lc \\
&\frac{dL}{dt} = nL \\
&K(0)>0 \;\; L(0)>0.
\end{aligned} \tag{2}$$

The control variable is c and the state variables are K and L. The analytic solution of this model yields one saddle point stationary state for per capita, consumption, and utility in the economy to the left of the maximum consumption point. The comparative statics of the system in the neighborhood of the stationary state are unambiguous:
$\partial c/\partial b < 0$, $\partial c/\partial n < 0$, $\partial c/\partial \rho < 0$; $\partial k/\partial b < 0$, $\partial k/\partial n < 0$, $\partial k/\partial \rho < 0$, where k=K/L; and $\partial U/\partial b < 0$, $\partial U/\partial n < 0$.

Incorporating exogenous labor-productivity enhancing technological change in the economy modifies the representative agent's maximization problem as follows:

$$\max \int_0^{\infty} U(c) e^{-\rho t} dt$$

$$\begin{aligned}
&s.t. \\
&\frac{dK}{dt} = F(K, AL) - \delta K - Lc \\
&\frac{dL}{dt} = nL \\
&\frac{dA}{dt} = \alpha A \\
&K(0)>0 \;\; L(0)>0.
\end{aligned} \tag{3}$$

A is the technology parameter that grows exogenously at rate $\alpha$ . The comparative dynamic (short-run transition) effect of an increase in the rate of technology change $\alpha$ unambiguously increases per capita level of the variables for consumption, capital stock, and utility. In the long run, these variables grow at the constant rate of technological advance $\alpha$.

In summary, the systems dynamics of the standard neoclassical economics model are simple and unambiguous. Of particular note, both the short and long run effects of technology change are unambiguously positive.

The models we study all represent minor variations on the standard economic growth models described in (2) and (3). In the first two variations we study an economy dependent on natural resource capital, rather than physical capital. We denote nature capital as S. Nature capital in this macro context is construed broadly as a generic renewable resource, in the manner of other macro-level studies (Maler, 1991; Brander and Taylor, 1998; Dasgupta and Maler, 2000). Having an economy whose output is entirely produced by labor and natural resources is a stylization that allows us to isolate for the dynamic systems effects of natural resources in the macroeconomy. The first modification is to expand the utility functions described in 2 and 3 to allow for the fact that representative agent may derive direct utility from nature capital, as well as direct consumption. The utility specification represents a broadened quality of life measure which, at the most primitive level, reflects the possible utility trade-off implicit in the distinction between "growth" and "development" as these terms have been distinguished in the sustainable development literature. The analogue formulation to (2) with this modification is as follows:

$$
\begin{aligned}
&\max\int_0^\infty [U(C/L)+V(S/L)]e^{-\rho t}dt\\
&s.t.\\
&\frac{dS}{dt}=G(S,L)-bS-C\\
&\frac{dL}{dt}=nL\\
&S(0)>0,\ L(0)>0;\ S(t)>0\ \ C(t)\geq 0,\ \forall\ t
\end{aligned}
\tag{4}
$$

The control variable is $c \equiv C/L$, and the state variables are S and L. All the functional form assumptions stated before continue to hold.[1] Note that dS/dt, the resource accumulation equation, has the same functional properties as the standard logistic equation used in the resource economics literature when L is taken as a parameter. In the context of this model, factor input -- L in our case -- can the increase the productivity of nature capital (see Maler, 1991; and Dasgupta and Maler, 2000). Nature capital augmentation can be construed quite broadly, e.g., as the recovery of a degraded resource, the improved management of an existing resource, or the discovery of a new resource.

Notice that (4) is mathematically equivalent to (2) in every respect except except that the utility measure has been broadened to allow for utility from nature capital, V(S/L) as well per capita consumption (C/L). The mathematical equivalence except for one feature allows us to isolate for dynamic effects of broadening the quality of life measure alone within the conventional growth modeling framework. The analogue equation to (3) for this modification is as follows:

$$
\begin{aligned}
&\max\int_0^\infty [U(C/L)+V(S/L)]\exp(-\rho t)dt\\
&s.t.\\
&\frac{dS}{dt}=G(S,AL)-bS-C\\
&\frac{dL}{dt}=nL\\
&\frac{dA}{dt}=\alpha A\\
&S(0)>0,\ L(0)>0\ \ A(0)>0;\ S(t)>0,\ C(t)\geq 0,\ \forall\ t
\end{aligned}
\tag{5}
$$

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[1] It is also assumed in all of the models we study that property rights are fully specified and costlessly enforced. This is to maintain comparability with Ramsey construct. Relaxing the assumption would strengthen the conclusions reached in this paper, by complicating systems dynamics still further.,

The broadened utility specifications in (4) and (5) represent a structural modification on the demand side of the economy. We now consider a modification on the supply side of the supply side of the economy. Notice that equations of motion for dK/dt in (1) presumes that consumption is "free" in the sense that there is no cost except the opportunity cost of reduced capital accumulation. It is more reasonable to assume, particularly in the natural resource cost, that C imposes a direct cost as well. Effort is expended to extract oil, or pump water for example. Even in a neoclassical economy, a transactions or transit cost might be associated with the distribution of C to the final market. Thus, our next model adapts the structure of motion for capital in (1)-(5) to the following:

$$\frac{dS}{dt}=G(S,L_1)-bS-C(S,L_2). \tag{6}$$

Nature capital consumption is represented as $C(S,L_2)$ where $L_2$ is the labor allocation to harvesting or extraction, with $C_s>0$, $C_{ss}<0$, $C_{L2}>0$, $C_{L2L2}<0$, and $C_{SL2}\geq 0$, and $C(0,L_2)=C(S,0)=0$. This channel is an important aspect of the resource-based production process represented in the resource literature, e.g., Clark (1976), Munro and Scott (1985), and Hanley et. al. (1997). Going back to the standard utility formulation to isolate for the dynamic impact of changing the supply side of the economy, the agent's maximization problem in this case can be formulated as follows:

$$\begin{aligned}
&\max\int_0^\infty U[C(S,L_2)/L]e^{-\rho t}dt\\
&s.t.\\
&\frac{dS}{dt}=G(S,L_1)-bS-C(S,L_2)\\
&\frac{dL}{dt}=nL\\
&L=L_1+L_2\\
&S(0)>0\ \ L(0)>0
\end{aligned} \tag{7}$$

In this context, $L_1$ and $L_2$ are the control variables and S and L are the state variables. The representative agent is allocating labor between nature capital augmentation and nature capital extraction at each instant, in order to maximize the sum of his/her discounted utilities over an infinite horizon. This formulation essentially isolates the effects of resource extraction costs within the standard Ramsey economic growth framework, since the model otherwise shares the same dynamic structure as the standard formulation.

Equation (7) with technology change can be formulated as follows:

$$
\begin{aligned}
&\max \int_0^{\infty} U(C/L)e^{-\rho t}dt \\
&s.t. \\
&\frac{dS}{dt} = G(S, AL_1) - bS - C \\
&\frac{dL}{dt} = nL \\
&\frac{dA}{dt} = \alpha A \\
&L = L_1 + L_2 \\
&C = C(S, AL_2) \\
&A(0) = A_o = 1 \;\; S(0)>0 \;\; L(0)>0
\end{aligned}
\tag{8}
$$

As mentioned, the labor force exogenously evolves in the standard neoclassical growth model, and in all of the modifications described heretofore in this section. Exogenous population growth is clearly an unrealistic assumption. Thus, our final modification is to go back to the standard Ramsey formulation in equation (2), and relax the assumption that population grows exogenously. This is effectively maintaining the demand and supply side of the neoclassical economy, but modifying a feedback relationship within it. This modified model can be described as follows:

$$
\begin{aligned}
&\max \int_0^{\infty} U(c)e^{-\rho t}dt \\
&s.t. \\
&\frac{dK}{dt} = F(K, L) - \delta K - Lc \\
&\frac{dL}{dt} = n[f(K/L)] \cdot L \\
&K(0)>0 \;\; L(0)>0.
\end{aligned}
\tag{9}
$$

The point of departure from (2) is defining n(f(k)), i.e., n, the population growth rate, depends explicitly on per capita output (or income). The first and second derivatives of n(f(k)) are left unspecified in the general case, to allow for the many different possibilities described in the demography literature.

In summary, the models of this section all represent minor departures from the mathematical structure of the standard neoclassical growth model reported in (2), to determine the dynamic consequences of specific, and reasonable, departures from idealized neo-classical simplifying assumptions on the demand and supply sides of the economy, and in feed back relationships within the economy. While these departures from the standard model are all minor, we will see that they have profound implications for the models' dynamic solution.

We turn the to the analytic solution of these models the following sections. Section 3 considers the steady-states and stability properties in the systems. Section 4 considers comparative static effects, while section 5 discusses technology change.

## 3. Steady States and Stability

Figures 1, 3, and 3 show the basic phase diagrams underlying the analytic solutions of the models in equations (4), (7) and (9). Figure 1 shows the solution of (4): the impact of broadening the quality of life representation. Equilibria 1 and 4 are saddle point stable; Equilibrium 2 is an unstable node, while Equilibria 3 is a non-isolated critical point that has one stable arm in one quadrant, and an unstable arm in each of the three other quadrants. The system in Figure 1 will collapse to an alternating sequence of saddle points and unstable nodes (equilibria 1, 2, and 4) if a tangency point does not occur. However, the tangency condition cannot be mathematically ruled out at the level of general functional forms. This can be contrasted with the analytic solution to equation (2), the neoclassical economic growth model, which would have one saddle point stable equilibrium the left of the maximum consumption point.

In terms of the solution of the model in which the utilization of nature capital imposes a direct cost, i..e, the model in (7) above, the number of steady states in the system is undefined. There could be one, more than one, or none at all. However, if there are steady states, they will always be saddle point stable. Figure 2 reflects this configuration. Note that the phase diagram is in s-$\theta$ space, where s is per capita resource stock and $\theta$ is labor share in nature capital extraction. Since $c(s,\theta)$, the phase diagram shows different equilibria for per capita consumption and utility levels.

Finally, the phase diagrams for cases of endogenous fertility are shown in Figures 3 and 4. Multiple equilibria are possible in this system, and always alternate between saddle points and unstable nodes. The first equilibria may be either a unstable node or a saddle point stationary state. The rest of the sequence follows accordingly (Figure 3 and 4 reflect the two alternative start-points).

Multiple equilibria in these systems raise the question of systems resiliency discussed in the ecological economics literature. Consider Figure 2, for example, and imagine a horizontal line drawn through equilibrium 2, extending leftward to the $\theta$ axis and rightward to infinity. The leftward extension represents the initial points of a negative shock to nature capital (s), and the rightward extension represents the initial points of a positive shock to s, holding the initial sectoral labor share constant.[2] Along this line, there is a "domain of attraction" around

[2] Points further to the left (right) indicate more negative (positive) shocks.

equilibrium 2, as this term is used in the ecosystem reliance literature (see, e.g., Ludwig et al., 1997). Within that domain, the system could respond to shocks to s by returning to equilibrium 2. If a negative shock is large enough to move the resource outside of this domain, the system would be more likely to move to equilibrium 1.[3] While inside this domain of attraction, the system would be "resilient" with reference to resource shocks around equilibrium 2.

**4. Comparative static effects of parametric changes.**

The differences in the comparative statics between models in (2) and (4) are shown in Table 1. Differences emerge through two channels. The model with nature capital in utility can exhibit one saddle point stable equilibrium on the nature capital domain to the right of the maximum consumption point, and multiple equilibria of three types on the domain to the left of the maximum consumption point. In the range beyond the maximum consumption point, the comparative static effects of changes in $\delta$ and b on consumption are ambiguous in the model with nature capital in utility, in contrast to the Ramsey model, where consumption initially declines with increases in n or b for the single saddle point occurring left of the maximum consumption point. The comparative statics of the social rate of discount also are different from the standard case. Consumption initially increases with a rise in the discount rate for the saddle point beyond the maximum consumption point, in contrast to the standard model, where consumption declines.

Around unstable nodes – which do not exist in the standard Ramsey model -- the expanded model with nature capital in utility manifests the seemingly perverse result that consumption and nature capital initially rise with population growth rate, capital depreciation, and discount rate. The differences in the welfare effects of parametric changes between the model with nature capital in utility and the standard Ramsey model reflect both the differences in the comparative static effects on nature capital and consumption just noted, and the fact that the expanded model has a two-argument utility function, with utility monotonically increasing in both consumption and nature capital.

The comparative static effects in model 7– the costly extraction model – simply turn out to undefined in every instance save one. In that case, an increase in a parameter for extraction efficiency in a Cobb-Douglas economy actually lowers welfare. Thus, technological progress has the opposite impact in this model than in the standard Ramsey model.[4]

In all, these results demonstrate that a relatively small departure from the conventional assumptions in the economic growth literature can dramatically alter the initial system responses to changes in the system's parameters.

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[3] A large positive shock may move the system from equilibrium 2 to equilibrium 3.

[4] We did not explore comparative statics or technology progress in the model of endogenous fertility, so our discussion for the remainder of this paper deals with the demand and supply side modifications.

## 5. Technology Change

The comparative dynamic and long-run effects of technology change for the broadened utility model are compared to the standard Ramsey model in Table 2. The different technology effects are quite striking, suggesting that even technology change that would seem presumptively propitious -- technology increasing exogenously at an exponential rate -- could exert negative short-run welfare effects in realistically specified models. Since the "short-run" is not clearly defined by theory, the possibility cannot be ruled out that negative technology effects could persist for a relatively "long period" benchmarked against a typical planning horizon for humans.

The multiple equilibria structure of the static model translates into the possibility of multiple balanced growth paths in the model with broadened quality of life measure. That implies that economies with the same parameter configuration may not necessarily converge to the same balanced growth path – as they must in a neoclassical growth model.

The results of technology in the model with costly nature capital extraction are qualitatively similar in all respects save 1. Unlike both models displayed in Table 2, the economy with costly nature capital extraction will grow at long run rate less than the rate of technologic progress. Hence, even the long-run effect of technology is less propitious than in the standard Ramsey model.

## 6. Conclusions

Multiple equilibria are common in ecological models (see, e.g., May, 1977; Ludwig, et al., 1997). The associated dynamic complexity of such models and its economic implications have become the focus of an emerging literature in ecological economics (e.g., Perrings and Walker, 1997; Arrow et al., 2000; Maler, 2000). Although the concept of multiple equilibria is sometimes used in economics (e.g., in game theoretic models), the issue has not received much attention in the mainstream economic growth literature, nor in its endogenous technological progress and environmental extensions. Our paper shows that the system dynamics of realistically modified growth models turn out to be remarkably complex. This suggests the benefit of bringing a more "systems ecological" perspective to the task of economic growth modeling, in the sense of making systems dynamics a primary modeling focus. Within this context, the "ecological economics" terminology could be construed as the notion that the principles of systems modeling in ecology could be usefully extended to the study of economic growth. This task could prove to be fruitful in view of the possibility that the dynamic complexity in the economic component of the system would reinforce the dynamic complexity injected into the system by its ecological component.

Presumptively, dynamic diversity in the macroeconomy has policy implications. For example, the accuracy of estimates generated by conventional computable general equilibrium models of the economic effects of global climate change are questionable, since these models, in the main, rely on the traditional Ramsey growth framework for their theoretical foundation (e.g., see Jorgenson and Wilcoxen, 1991). The existence of multiple growth paths adds another dimension to the claim that the criterion of non-declining welfare alone, which is sometimes cited as an indicator for sustainable development, is not a sufficient policy-making criterion (see, e.g., Toman et. al., 1995). With more than one sustainable path, some other criteria would have to used to choose among them. Another important implication of multiple growth paths is that differences in initial conditions, reflecting historical accident or institutional factors, could persist in the form of long-run disparities in welfare levels, regardless of global technical advance. This in turn suggests a greater role for policy-making to achieve desired welfare outcomes. Freely-traded technology that causes the global rate of technology advance to converge across countries may not be sufficient to cause a convergence in welfare levels. Wealth transfers from rich to poor countries may be needed to encourage a higher welfare, sustained development path of poorer economies.

## References

Arrow K., G. Daily, P. Dasgupta, S. Levin, K-G Maler, E. Maskin, D. Starret, T Sterner, and T. Tietenberg, Managing ecosystem resources, *Environmental Science and Technology* 34, 1401-1406 (2000).

Aghion, P. and P. Howitt, *Endogenous Growth Theory,* MIT Press: Cambridge, MA (1998).

Brander, J. A. and M. S. Taylor, The simple economics of Easter Island: a Ricardo-Malthus model of renewable resource use, *American Economic Review* 88, 119-138 (1998).

The Brundtland Commission, *World Commission on Environment and Development: Our Common Future*, Oxford University Press and United Nations: New York (1987).

Clark, C., *Mathematical Bioeconomics: The Optimal Management of Renewable Resources,* New York: John Wiley and Sons, Inc. (1990).

Clark, C. and G. Munro, The economics of modern capital theory: A simplified approach, in L. J. Mirman and D. F. Spulber (eds), Essays in the Economics of Renewable Resources, Amsterdam: North Holland (1982).

Daly, H. E., *Steady State Economics: Second Edition with New Essays*, Earthscan: London (1992).

Dasgupta P. and Maler, K. G., Net national product, wealth, and social well being, *Environment and Development Economics* 5, 1-25 (2000).

Hartwick, J.M. and N.D. Olewiler, *The Economics of Natural Resource Use*, New York: Harper & Row (1986).

Jorgenson D.W. and P.J. Wilcoxen, Reducing US carbon dioxide emissions: the cost of different goals. In *Energy, Growth, and the Environment*, ed. J.R., Moronoy, Greenwich JAI: CT (1991).

Krutilla and Reuveny, The quality of life in the dynamics of economic development, forthcoming, *Environment and Development Economics* (2001a).

Krutilla, K. and R. Reuveny, A renewable resource-based model with extraction costs, *Working Paper*, School of Public and Environmental Affairs, Indiana University, Bloomington, IN (2001b).

Krutilla, K. and R. Reuveny, The effects of endogenous population growth in a Ramsey economic growth model, *Working Paper*, School of Public and Environmental Affairs, Indiana University, Bloomington, IN (2001c).

Ludwig, D., B. Walker and C. S. Holling, Sustainability, stability, and resilience, *Conservation Ecology* [online] 1, URL: http://www.consecol.org/vol1/iss1/art7 (1997).

Maler, K-G., National accounts and environmental resources, *Environmental and Resource Economics* 1, 1-15 (1991).

Maler, K. G., Development, ecological resources and their management: A study of complex dynamic systems, *European Economic Review* 44, 645-665 (2000).

May, R. M., Thresholds and breakpoints in ecosystems with multiple steady states, *Nature* 269, 471-477 (1977).

Munro, G. and Scott, A., The economics of fisheries management, in *Handbook of Natural Resource and Energy Economics*, *Volume II*, (A. Kneese and J. L. Sweeney, eds.), Amsterdam: North Holland (1985).

Perrings, C. and B. Walker, Biodiversity, resilience and the control of ecological-economic systems: The case of fire-driven rangelands, *Ecological Economics* 22, 73-83 (1997).

Stiglitz, J., Growth with exhaustible natural resources: Efficient and optimal growth path, *Review of Economic Studies*, *Symposium on Natural Resources*, 123-137 (1974).

Stern, D.I., and M.S. Common, Is there an environmental Kuznets curve for sulfur? *Journal of Environmental Economics and Management* 41(4): 162-178 (2001).

Toman, M. A., J. Pezzey and J. Krautkraemer, Neoclassical economic growth theory and sustainability, in Daniel W. Bromley (ed.) *The Handbook of Environmental Economics*, Blackwell: Cambridge, MA (1995).

World Bank, Expanding the measure of wealth: Indicators of environmentally sustainable development, *Environmental Sustainable Development Studies, Monograph No. 17.* The World Bank (1997).

**Table 1***

**Ramsey Model versus Expanded Quality of Life Model, Static Technology**
**Summary of Comparative Static Effects**

| | Standard Ramsey Model | Model with Expanded Quality of Life Measure |
|---|---|---|
| **Around Saddle Points** | | |
| **$(0, s_{msy})$** | | |
| **Consumption** | $\partial c/\partial b < 0$, $\partial c/\partial n < 0$, $\partial c/\partial \rho < 0$ | $\partial c/\partial b < 0$, $\partial c/\partial n < 0$, $\partial c/\partial \rho < 0$ |
| **Nature Capital** | $\partial s/\partial b < 0$, $\partial s \partial n < 0$, $\partial s/\partial \rho < 0$ | $\partial k/\partial b < 0$, $\partial s/\partial n < 0$, $\partial s/\partial \rho < 0$ |
| **Welfare** | $\partial W/\partial b < 0$, $\partial W/\partial n < 0$ | $\partial W/\partial b < 0$, $\partial W/\partial n < 0$ |
| **Around Saddle Points** | | |
| **$(s_{msy}, s^{+})$** | | |
| **Consumption** | NA** | $\partial c/\partial b < 0$ or $>0$, $\partial c/\partial n < 0$ or $>0$, $\partial c/\partial \rho > 0$ |
| **Capital** | NA | $\partial s/\partial b < 0$, $\partial s/\partial n < 0$, $\partial s/\partial \rho < 0$; |
| **Welfare** | NA | $\partial W/\partial b < 0$ or $>0$, $\partial W/\partial n < 0$ or $>0$ |
| **Around Unstable Nodes** | | |
| **Consumption** | NA | $\partial c/\partial b > 0$, $\partial c/\partial n > 0$, $\partial c/\partial \rho > 0$ |
| **Capital** | NA | $\partial s/\partial b > 0$, $\partial s/\partial n > 0$, $\partial s/\partial \rho > 0$ |
| **Welfare** | NA | $\partial W/\partial b > 0$, $\partial W/\partial n > 0$ |
| **Around Non-Isolated Critical Points** | | Undefined |

* results derived in the appendix
** NA=not applicable

## Table 2

**Ramsey Model versus Expanded Quality of Life Model, Technological Progress**

| | **Standard Ramsey Model** | **Expanded Quality of Life Model** |
|---|---|---|
| **Balanced Growth Path** | Single | Single<br>Multiple |
| **Transition Path** | Smooth rise in c,s, and W | Initial decrease in s<br>Initial increase or decrease in c<br>Initial increase or decrease in W |
| **Long-Run Welfare** | Rate of technological progress | Rate of technological progress |
| **Growth Rate** | | |

Figure 1

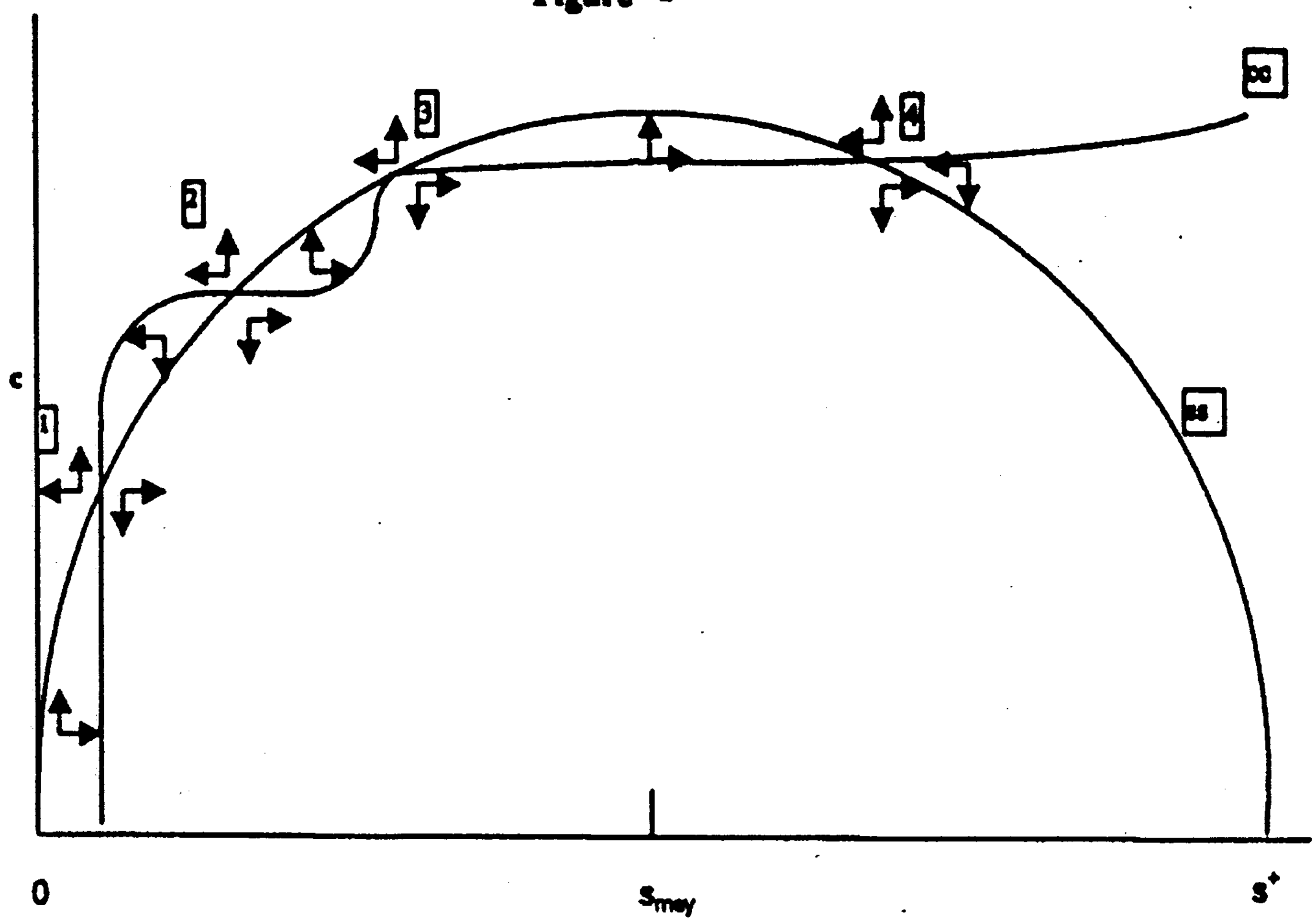

**Figure 2**

Phase Diagram

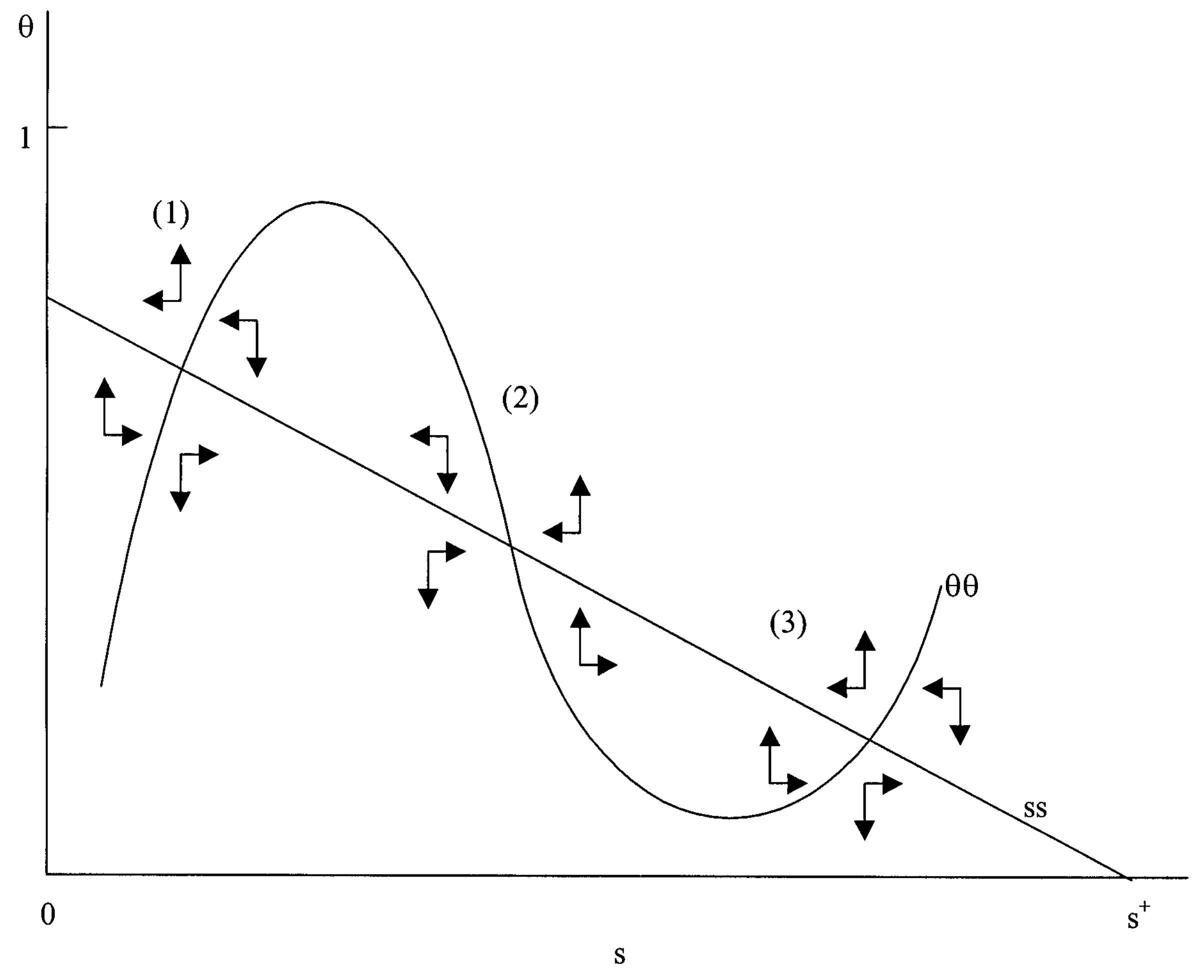

**Figure 3**
**Phase Diagram**

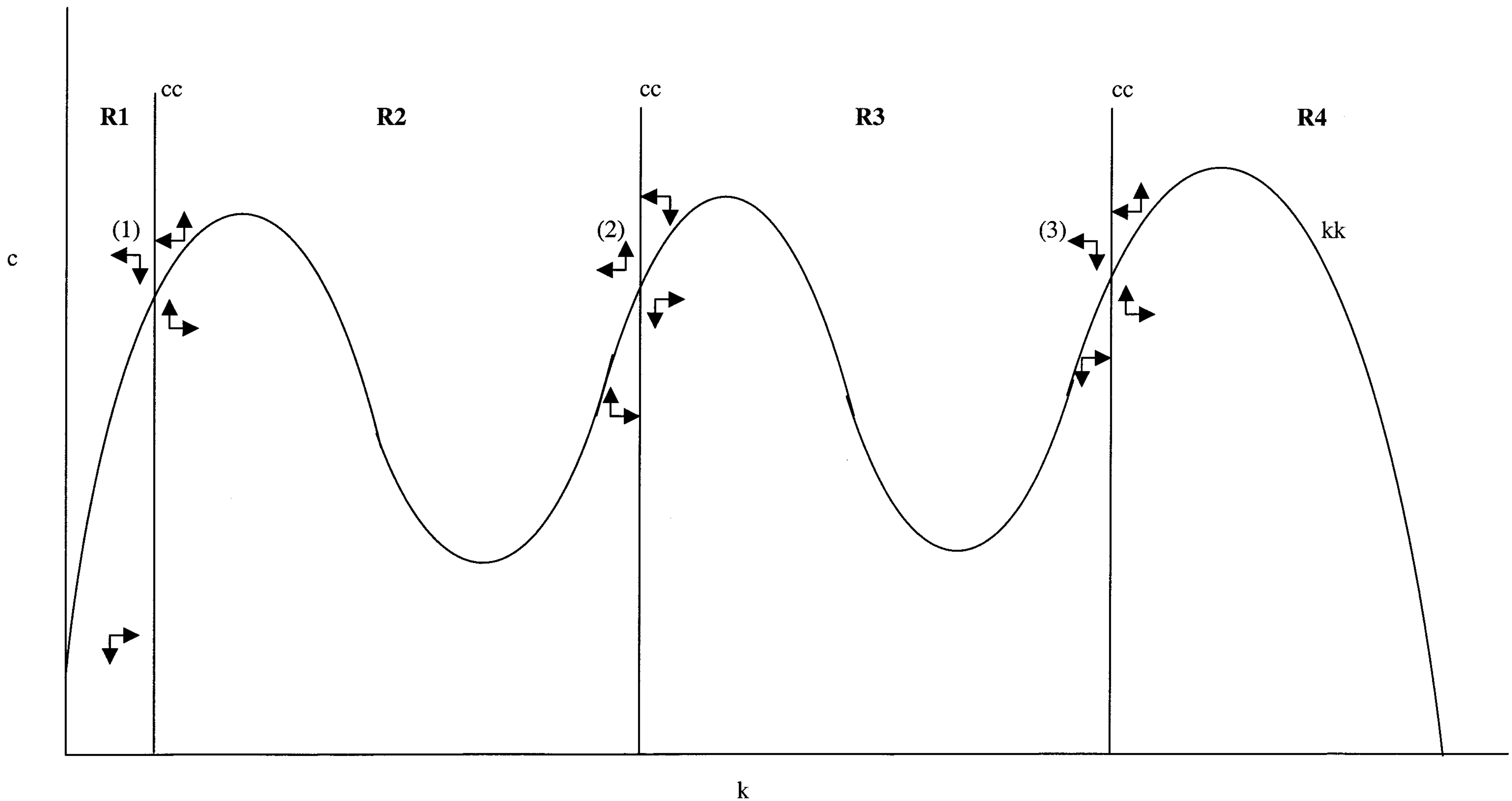

**Figure 4**
**Phase Diagram**